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Algorithm for Simulation of Motions of Variable-Mass Systems

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Variable-mass systems expel and/or capture particles during motion. A new algorithm for simulation of motions of such systems is introduced. Accordingly, the effect of the expulsion and/or capture of particles on the motion of the system is presented as changes in the integration variables of the governing dynamical equations. In connection with numerical solutions of the motion equations, the indicated changes are evaluated at each integration step. In contrast, conventional methods dealing with variable-mass systems give rise to dm/dt * $\boldsymbol{v}$-type terms in the motion equations, terms that account for particle expulsion and/or capture. The new formulation is developed for simple, nonholonomic systems and, with reference to an example, is shown to lead to results satisfying momentum principles. Furthermore, two kinds of systems are identified: continuous-particle-ejecting systems, such as rockets; and discrete-particle-ejecting systems, such as automatic weapons (that fire rounds, one at a time). In connection with these systems, conditions are specified, the satisfaction of which permits the use of the force-replacement approach to variable-mass systems on the one hand, and of the new formulation on the other.

Introduction

VARIABLE-MASS systems form a category in the field of dynamics dealt with by numerous authors. Such systems expel and/or capture particles while in motion. An accurate description of the chain of events leading to particle expulsion and/or capture requires a distinction between expulsion points (capture points), which are points from (at) which particles are expelled (captured), and rest points, which are points occupied by the particles before (after) being expelled (captured). It frequently occurs that these points do not coincide.

Authors dealing with variable-mass systems adopt one of three approaches: control volume, particle, or force replacement. The control-volume approach involves the definition of a control volume in connection with the system of interest, and the writing of motion equations for the material within that volume, with the aid of the linear-momentum and the angular-momentum principles. This approach underlies the works of such researchers as Meirovitch,[1] McPhee and Dubey,[2] and Eke and Wang.[3] It is advanced by classical textbooks[4] and is practically limited to the analysis of systems consisting of a single body. The particle approach focuses on inertial forces, $-m\boldsymbol{a}$-type quantities, associated with the expelled and/or captured particles, replacing these with $-m\boldsymbol{a} + \dot{m}\boldsymbol{v}$. This approach is taken, for example, by Ge and Cheng,[5] who, defining generalized thrust, extend Kane's equations to deal with simple, nonholonomic systems expelling particles; and by Ibrahim and Modi,[6] who use Lagrange equations with Lagrange multipliers and formulate motion equations for systems including deformable bodies, possibly undergoing constrained motions. Eventually, these authors arrive at equations containing $\dot{m}\boldsymbol{v}$-type quantities.

The force-replacement approach regards the $\dot{m}\boldsymbol{v}$-type quantities as contact forces (sometimes called thrust forces) exerted on expulsion points and/or on capture points and treats these forces accordingly. This approach is taken, for example, by Greensite[7] investigating space-vehicle flight and by Cveticanin.[8,9] The latter generates motion equations based on the Lagrange equation of the first kind, referring specifically to a quantity ϕ, which he calls "reactive force due to mass variation." Thus, analyses based on the force-replacement approach give rise to equations identical to those generated with the aid of the particle approach. However, there is a subtle difference between the two approaches, related to the fact that forces associated with expulsion and/or capture of particles frequently are known, e.g., through measurements. In connection with the force-replacement approach, such forces are allowed to replace $\dot{m}\boldsymbol{v}$-type quantities, and hence are treated similarly to other contact forces. Moreover, these (measured) forces take into account the effect of internal particle flow, an effect disregarded by users of the particle approach.

A new algorithm for simulation of motions of variable-mass systems is presented in the following sections. The underlying idea is that expulsion (capture) of particles is regarded as a continuous (or, in connection with numerical solutions, discretized) process of removal (imposition) of those constraints holding the particles in question to their rest points. Furthermore, these particles are excluded from the system after (before) being ejected (arrested), so that the number of equations governing the motion of the system remains unaltered. The new formulation differs from previous approaches in that it deals with simple, nonholonomic systems; it makes the distinction between expulsion point (capture point) and rest point, and it presents the effect of the expulsion and/or capture of particles as changes in the integration variables, changes that, in connection with numerical solutions of the motion equations, are evaluated at each integration step.

The paper is organized as follows. The new algorithm is developed, and, with reference to an example, is shown to lead to results satisfying momentum principles. Next, the new algorithm is discussed with reference to continuous-particle-ejecting systems such as rockets and to discrete-particle-ejecting systems such as automatic weapons that fire rounds one at a time. Finally, the force-replacement approach is compared with the new formulation when applied to the indicated systems. First, however, the theory of imposition and removal of constraints,[10] which underlies the new formulation, is reviewed briefly.

Theory of Imposition and Removal of Constraints: Main Results

Let S be a simple nonholonomic system of ν particles P_i ($i = 1, \ldots, \nu$) of mass m_i possessing $\bar{n}$ generalized coordinates $q_1, \ldots, q_{\bar{n}}$ and n (where $n < \bar{n}$) generalized speeds $u_1, \ldots, u_n$ in N. Moreover, let S undergo three phases of motion as follows: Phase 1 occurs in the time interval of $0 \leq t \leq t_1$. The motion of S in N is defined as unconstrained and is governed by n dynamical equations, namely,

$$F_r + F_r^* = 0 \qquad (r = 1, \ldots, n) \tag{1}$$

where F_r and F_r^* are the rth generalized active force and the rth generalized inertial force for S, respectively.

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Phase 2 occurs in the time interval $t_1 \le t \le t_2$, where $t_2 - t_1$ is infinitely small, e.g., compared with time constants associated with the motion of S. During this phase, m constraints of the form

$$u_k = \sum_{r=1}^{p} C_{kr} u_r + D_k \qquad (k = p+1, \ldots, n) \tag{2}$$

are imposed on S, where

$$p \mathrel{\hat{=}} n - m \tag{3}$$

and C_{kr} and D_k are functions of $q_1, \ldots, q_{\bar{n}}$ and time t. The configuration of S in N remains unaltered, that is,

$$q_r(t_2) = q_r(t_1) \qquad (r = 1, \ldots, \bar{n}) \tag{4}$$

and the number of independent generalized speeds is reduced from n to p. The relationship between $u_k(t_2)$ $(k = p+1, \ldots, n)$, the values of the dependent generalized speeds at t_2, and $u_r(t_2)$ $(r = 1, \ldots, p)$, the values of the independent generalized speeds at t_2, is given by

$$u_k(t_2) = \sum_{r=1}^{p} C_{kr} u_r(t_2) + D_k \qquad (k = p+1, \ldots, n) \tag{5}$$

Additionally, if the magnitudes of the active forces contributing to Eqs. (1) are all bounded, and if points of S exert contact forces on one another, and, possibly, on particles whose motion is not affected by the forces exerted on them by particles of S, then p relations between $u_s(t_2)$ $(s = 1, \ldots, n)$ and $u_s(t_1)$ $(s = 1, \ldots, n)$ are given by

$$\sum_{s=1}^{n} \left(m_{rs} + \sum_{k=p+1}^{n} C_{kr} m_{ks} \right) [u_s(t_2) - u_s(t_1)] = 0 \qquad (r = 1, \ldots, p) \tag{6}$$

Here, m_{rs}, the element in row r, column s, of the mass matrix associated with Eqs. (1), is defined as

$$m_{rs} \mathrel{\hat{=}} -\sum_{i=1}^{\nu} m_i \frac{\partial \boldsymbol{v}^{P_i}}{\partial u_r} \frac{\partial \boldsymbol{v}^{P_i}}{\partial u_s} \qquad (r, s = 1, \ldots, n) \tag{7}$$

where $\boldsymbol{v}^{P_i}$ is the velocity of P_i in N. Equations (5) and (6) furnish $m+p$ relations between $u_s(t_2)$ and $u_s(t_1)$ $(s = 1, \ldots, n)$ that enable one to evaluate the former, given the latter, with C_{kr}, D_k $(k = p+1, \ldots, n;\ r = 1, \ldots, p)$ and m_{rs} $(r, s = 1, \ldots, n)$ calculated at t_1.

Phase 3 occurs when $t > t_2$. Then, the motion of S in N is defined as constrained and is governed by p dynamical equations, namely,

$$F_r + F_r^* + \sum_{k=p+1}^{n} C_{kr} \left(F_k + F_k^* \right) = 0 \qquad (r = 1, \ldots, p) \tag{8}$$

It then is said that the constraints in Eqs. (2) are imposed on the motion of S, and the process is called imposition of constraints.

It may occur that Eqs. (2) are satisfied at $t = t_1$; that is,

$$u_k(t_1) = \sum_{r=1}^{p} C_{kr} u_r(t_1) + D_k \qquad (k = p+1, \ldots, n) \tag{9}$$

equations that, combined with Eqs. (5), lead to

$$u_k(t_2) - u_k(t_1) = \sum_{r=1}^{p} C_{kr} [u_r(t_2) - u_r(t_1)] \qquad (k = p+1, \ldots, n) \tag{10}$$

Equations (6) and (10) constitute a set of n homogeneous equations in n unknowns $u_s(t_2) - u_s(t_1)$ $(s = 1, \ldots, n)$, having the unique null solution, whereby

$$u_s(t_2) = u_s(t_1) \qquad (s = 1, \ldots, n) \tag{11}$$

Equations (11) imply that no impulsive forces arise; therefore, the process is called soft imposition of constraints. If, on the other hand, Eqs. (2) are not satisfied at $t = t_1$, then Eqs. (5) and (6) decree changes in the values of the generalized speeds, giving rise to impulsive forces; therefore, the process is called hard imposition of constraints.

When the chain of events just described occurs in reverse order, then Eqs. (8) govern phase 1. Phase 2 is governed by Eqs. (4), (6), and (9), and phase 3 is governed by Eqs. (1). It then is said that the constraints in Eqs. (2) are removed from S, and the process is called removal of constraints.

It may occur that $u_k(t_2)$ $(k = 1, \ldots, n)$ are interrelated as follows:

$$u_k(t_2) = \sum_{r=1}^{p} C_{kr} u_r(t_2) + D_k + E_k \qquad (k = p+1, \ldots, n) \tag{12}$$

where E_k $(k = p+1, \ldots, n)$ is a function of $q_1, \ldots, q_{\bar{n}}$ and t. If

$$E_k = 0 \qquad (k = p+1, \ldots, n) \tag{13}$$

then Eqs. (12) reduce to Eqs. (5), which, with Eqs. (9) and (6), yield Eqs. (10) and (11). Equations (11) imply that no impulsive forces arise; therefore, the process is called soft removal of constraints. If, on the other hand, Eqs. (13) are not satisfied, then Eqs. (6) and (12) decree changes in the values of the generalized speeds, giving rise to impulsive forces; therefore, the process is called hard removal of constraints.

Expulsion and/or Capture of Particles

Let $\bar{P}$ be a point of S, and $\bar{Q}$ be a particle of S coinciding with $\bar{P}$ both at t_1 and at t_2. Let ${}^N\boldsymbol{v}^{\bar{Q}}$ and ${}^N\boldsymbol{v}^{\bar{P}}$ denote the velocities of $\bar{Q}$ and $\bar{P}$ in N, respectively. Then, $\bar{Q}$ may undergo, in conjunction with $\bar{P}$, the following events:

Hard removal of constraints: $\bar{Q}$ and $\bar{P}$ coincide before t_1 and move independently after t_2 such that

$${}^N\boldsymbol{v}^{\bar{Q}}(t_1) = {}^N\boldsymbol{v}^{\bar{P}}(t_1) \tag{14}$$

$${}^N\boldsymbol{v}^{\bar{Q}}(t_2) = {}^N\boldsymbol{v}^{\bar{P}}(t_2) + {}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}(t_2) \tag{15}$$

Then, three constraints are removed from S, and $\bar{Q}$ is said to be ejected form $\bar{P}$. ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}(t_2)$ is called velocity of separation.

Soft removal of constraints: This is a special case of hard removal of constraints, whereby ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}(t_2) = 0$. Then, three constraints are removed from S, and $\bar{Q}$ is said to be released from $\bar{P}$.

Hard imposition of constraints: $\bar{Q}$ and $\bar{P}$ move independently before t_1 and coincide after t_2 such that

$${}^N\boldsymbol{v}^{\bar{Q}}(t_1) = {}^N\boldsymbol{v}^{\bar{P}}(t_1) + {}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}(t_1) \tag{16}$$

$${}^N\boldsymbol{v}^{\bar{Q}}(t_2) = {}^N\boldsymbol{v}^{\bar{P}}(t_2) \tag{17}$$

Then three constraints are imposed on S, and $\bar{Q}$ is said to be arrested by $\bar{P}$. ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}(t_1)$ is called velocity of approach.

Soft imposition of constraints: This is a special case of hard imposition of constraints, whereby ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}(t_1) = 0$. Then, three constraints are imposed on S, and $\bar{Q}$ is said to be attached to $\bar{P}$.

Analogous events, involving six constraints, can be defined in connection with a reference frame and a rigid body.

Suppose the equations governing motions of a simple, nonholonomic system S of p degrees of freedom possessing p (independent) generalized speeds $u_1, \ldots, u_p$, have been formulated without regard to the fact that particles are expelled (captured). Then, in connection with variable-mass systems, two cases are discussed (Fig. 1).

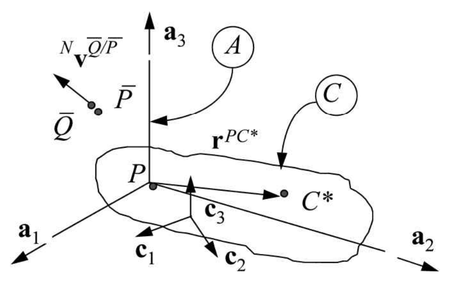

Fig. 1 Particles released from C and ejected from $\bar{P}$.

1) Particles yet to be expelled are arranged as a rigid body C fixed in a reference frame A belonging to S. Let $\boldsymbol{c}_i$ and $\boldsymbol{a}_i$ $(i = 1, 2, 3)$ be two sets of three dextral, mutually perpendicular unit vectors fixed in C and A, respectively, defined so that $\boldsymbol{c}_i$ $(i = 1, 2, 3)$ are parallel to central principal axes of C. Let P and $\bar{P}$ be points of S, and let P be fixed in A. Finally, let m_C and $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i$ $(i = 1, 2, 3)$ be, respectively, the mass of C, and the measure numbers of the position vector from P to C^*, the mass center of C; and let I_i^C and C_{ij} $(i, j = 1, 2, 3)$ be, respectively, the central principal moments inertia of C, and the direction cosines between $\boldsymbol{a}_i$ and $\boldsymbol{c}_j$ $(i, j = 1, 2, 3)$. Then the assumptions associated with this case are that particles are released from C and, simultaneously, attached to $\bar{P}$ and ejected from $\bar{P}$ with a relative velocity ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}$, $\bar{Q}$ being a typical particle of C ejected from $\bar{P}$; and that the manner in which particles arrive from points they occupy as parts of C—rest points—to $\bar{P}$, from which they are eventually ejected, is immaterial. That is, the effect of the internal particle flow is disregarded. (The discussion of ad hoc provisions taking these effects into account[1] is beyond the scope of this work.) Finally, m_C, $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i$, I_i^C, and C_{ij} $(i, j = 1, 2, 3)$, are assumed to be known functions of time.

2) Particles yet to be captured impact point $\bar{P}$ of S with a relative velocity ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}$, $\bar{Q}$ being a typical particle captured at $\bar{P}$. It is assumed that these particles build up a rigid body C fixed in a reference frame A of S, and that, as before, m_C, $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i$, I_i^C, and C_{ij} $(i, j = 1, 2, 3)$ are known functions of time.

Let Δt be the time step associated with numerical solutions of the problem at hand. Let t be the current time, and define instants t_1 and t_2 as

$$t_1 \hat{=} t + \Delta t - (\delta/2), \qquad t_2 \hat{=} t + \Delta t + (\delta/2) \tag{18}$$

so that $\delta = t_2 - t_1$ is infinitely small. With reference to case 1, let

$$m_C(t_2) = m_C(t_1) - \frac{\mathrm{d}}{\mathrm{d}t} m_C \bigg|_{t_1} \Delta t \tag{19}$$

$$\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i|_{t_2} = \boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i|_{t_1} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i\right)\bigg|_{t_1} \Delta t \qquad (i = 1, 2, 3) \tag{20}$$

$$I_i^C(t_2) = I_i^C(t_1) - \frac{\mathrm{d}}{\mathrm{d}t} I_i^C \bigg|_{t_1} \Delta t \qquad (i = 1, 2, 3) \tag{21}$$

$$C_{ij}(t_2) = C_{ij}(t_1) + \frac{\mathrm{d}}{\mathrm{d}t} C_{ij} \bigg|_{t_1} \Delta t \qquad (i = 1, 2, 3) \tag{22}$$

Define Δm as $(\mathrm{d}m_C/\mathrm{d}t)\Delta t$, and let C' be a rigid body of mass Δm fixed in C as shown in Fig. 2. Let C' be released from C between t_1 and t_2, and simultaneously, let a particle $\bar{Q}$, now assumed to be of mass Δm, be attached to $\bar{P}$ and ejected from $\bar{P}$ with relative velocity ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}$. (One may conceive the ejection of a rigid body rather than the ejection of a particle. However, such a state of affairs rarely represents actual systems and is not pursued here.) Then, the equations governing the motion of S when $t \le t_1$ and when $t \ge t_2$ are

$$F_r + F_r^* + \left(F_r + F_r^*\right)' = 0 \qquad (r = 1, \ldots, p) \tag{23}$$

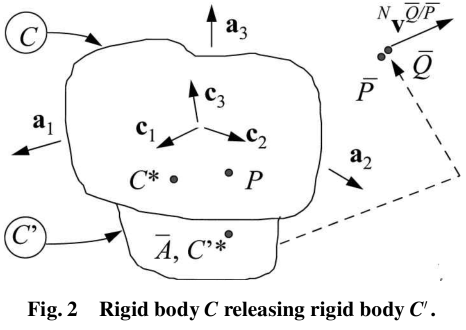

Fig. 2 Rigid body C releasing rigid body C'.

and

$$F_r + F_r^* = 0 \qquad (r = 1, \ldots, p + 3) \tag{24}$$

respectively, where, for $r = 1, \ldots, p$, $F_r + F_r^*$ are the contributions to the sum of the rth generalized active forces and the rth generalized inertial force from S (including C), and $(F_r + F_r^*)'$ are similar contributions from C'; and where the last three of Eqs. (24) are associated with the motion of $\bar{Q}$ after being ejected, and are decoupled from the first p equations. Moreover, note that because $\bar{P}$ belongs to S, the velocity of $\bar{P}$ in N can be expressed[11] as

$${}^N\boldsymbol{v}^{\bar{P}} = \sum_{r=1}^{p} {}^N\boldsymbol{v}_r^{\bar{P}} u_r + {}^N\boldsymbol{v}_t^{\bar{P}} \tag{25}$$

where ${}^N\boldsymbol{v}_r^{\bar{P}}$ and ${}^N\boldsymbol{v}_t^{\bar{P}}$ are functions of $q_1, \ldots, q_{\bar{n}}$ and t. Define $3p$ quantities $\bar{C}_{kr}$ as

$$\bar{C}_{kr} \hat{=} {}^N\boldsymbol{v}_r^{\bar{P}} \cdot \boldsymbol{d}_{k-p} \qquad (k = p + 1, p + 2, p + 3; r = 1, \ldots, p) \tag{26}$$

and, supposing ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}$ is known, define three quantities $\bar{E}_k$ as

$$\bar{E}_k \hat{=} {}^N\boldsymbol{v}^{\bar{Q}/\bar{P}} \cdot \boldsymbol{d}_{k-p} \qquad (k = p + 1, p + 2, p + 3) \tag{27}$$

where $\boldsymbol{d}_i$ $(i = 1, 2, 3)$ are three dextral, mutually perpendicular unit vectors fixed in reference frame D, chosen at will. Furthermore, identify the $p \times p$ mass matrix M of the p first of Eqs. (24) [having m_{rs} in Eqs. (7) as the element in row r, column s $(r, s, = 1, \ldots, p)$]; a $3 \times p$ matrix $\bar{C}$ having $\bar{C}_{kr}$ in Eqs. (26) as the element in row $(k - p)$, column r; a 3×1 matrix $\bar{E}$ having $\bar{E}_k$ in Eqs. (27) as its $(k - p)$th element; and a $p \times 1$ matrix u having u_r $(r = 1, \ldots, p)$ as its rth element. Then, the numerical values of the generalized speeds at t_2 are related to those at t_1 by the matrix equation

$$u(t_2) = u(t_1) + [M - \Delta m \bar{C}^T \bar{C}]^{-1} \Delta m \bar{C}^T \bar{E} \tag{28}$$

where superscripts T and -1 denote transposition and inversion, respectively. Equations (23), (24), and (28) govern the motion of S, and can be used repeatedly to simulate numerically an ongoing particle release–attachment–ejection process.

The following five-step procedure underlies the indicated simulation:

1) Set initial conditions; set initial values to the quantities appearing in Eqs. (19–22), namely, $m_C(0)$, $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i|_{t=0}$, $I_i^C(0)$, and $C_{ij}(0)$ $(i, j = 1, 2, 3)$.

2) Perform a one-step integration of Eqs. (23); denote the resulting values of u as $u(t_1)$, and the current values of the quantities appearing in Eqs. (19–22) as $m_C(t_1)$, $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i|_{t_1}$, $I_i^C(t_1)$, and $C_{ij}(t_1)$ $(i, j = 1, 2, 3)$, respectively.

3) Evaluate $m_C(t_2)$, $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i|_{t_2}$, $I_i^C(t_2)$, and $C_{ij}(t_2)$ $(i, j = 1, 2, 3)$ with the aid of Eqs. (19–22), and replace $m_C(t_1)$, $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i|_{t_1}$, $I_i^C(t_1)$, and $C_{ij}(t_1)$ $(i, j = 1, 2, 3)$ in Eqs. (23) with $m_C(t_2)$, $\boldsymbol{r}^{PC^*} \cdot \boldsymbol{a}_i|_{t_2}$, $I_i^C(t_2)$, and $C_{ij}(t_2)$ $(i, j = 1, 2, 3)$, respectively, obtaining the p first of Eqs. (24).

4) Solve Eq. (28) for $u(t_2)$.

5) If $m_C(t_2) = 0$, stop. Otherwise, rename the first p of Eqs. (24) as Eqs. (23), regard $u(t_2)$ as initial conditions, and return to step 2.

The validity of the five-step procedure, and specifically of Eqs. (23), (24), and (28), can be discussed after two points in time t' and t'' are introduced so that

$$t_1 < t' < t'' < t_2 \tag{29}$$

Let C' be released from C between t_1 and t', and transformed into $\bar{Q}$, which then is attached to $\bar{P}$ between t' and t''. These two events qualify as soft removal and soft imposition of constraints, respectively, and they involve no change in the values of the generalized speeds, so that at t'', when $\bar{Q}$ is attached to $\bar{P}$,

$$u_s(t'') = u_s(t_1) \qquad (s = 1, \ldots, p) \tag{30}$$

Next, let $\bar{Q}$ be ejected from $\bar{P}$ between t'' and t_2 with relative velocity ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}$, and note that the associated change in the inertial properties of S is brought into evidence by step 3. In connection with the motion of $\bar{Q}$ after t_2, define u_k $(k = p+1, p+2, p+3)$ as

$$u_k \triangleq {}^N\boldsymbol{v}^{\bar{Q}} \cdot \boldsymbol{n}_{k-p} \qquad (k = p+1, p+2, p+3) \tag{31}$$

so that

$${}^N\boldsymbol{v}^{\bar{Q}} = u_{p+1}\boldsymbol{n}_1 + u_{p+2}\boldsymbol{n}_2 + u_{p+3}\boldsymbol{n}_3 \tag{32}$$

Consequently, S—including $\bar{Q}$—can be regarded as undergoing an unconstrained motion governed by Eqs. (24). Moreover, it is implied by Eq. (32) that only $\bar{Q}$ contributes to F_r^* $(r = p+1, p+2, p+3)$. Specifically, $F_r^* = -\Delta m\dot{u}_r$ $(r = p+1, p+2, p+3)$. These expressions indicate that the first p of Eqs. (24) are decoupled from the last three equations. The latter, associated with the motion of $\bar{Q}$ after t_2, are of no interest, and can be disregarded. Steps 5 and 2 thus are justified.

Finally, consider the ejection of $\bar{Q}$ from $\bar{P}$ $(t'' \le t \le t_2)$. If substitutions are made from Eqs. (32) and (25) in Eqs. (14) and (15) with t'' playing the role of t_1, and the resulting equations are dot multiplied with $\boldsymbol{n}_{k-p}$ throughout, one has

$$u_k(t'') = \sum_{r=1}^{p} \bar{C}_{kr} u_r(t'') + \bar{D}_k \qquad (k = p+1, p+2, p+3) \tag{33}$$

$$u_k(t_2) = \sum_{r=1}^{p} \bar{C}_{kr} u_r(t_2) + \bar{D}_k + \bar{E}_k \qquad (k = p+1, p+2, p+3) \tag{34}$$

where

$$\bar{C}_{kr} \triangleq {}^N\boldsymbol{v}_r^{\bar{P}} \cdot \boldsymbol{n}_{k-p}, \qquad \bar{D}_k \triangleq {}^N\boldsymbol{v}_t^{\bar{P}} \cdot \boldsymbol{n}_{k-p}, \qquad \bar{E}_k \triangleq {}^N\boldsymbol{v}^{\bar{Q}/\bar{P}} \cdot \boldsymbol{n}_{k-p}$$
$$(k = p+1, p+2, p+3;\ r = 1, \ldots, p) \tag{35}$$

Here, bars are placed over C_{kr}, D_k, and E_k to accentuate their role as quantities associated with the velocities of the expulsion point and the ejected particle. Equations (34) are similar to Eqs. (12) for $k = p+1, p+2, p+3$ and can be used to evaluate $u_s(t_2)$ $(s = 1, \ldots, p)$ and $u_s(t_2)$ $(s = p+1, p+2, p+3)$ in conjunction with the equation

$$\sum_{s=1}^{p}\left(m_{rs} + \sum_{k=p+1}^{p+3} \bar{C}_{kr} m_{ks}\right)[u_s(t_2) - u_s(t'')]$$
$$+ \sum_{s=p+1}^{p+3}\left(m_{rs} \sum_{k=p+1}^{p+3} \bar{C}_{kr} m_{ks}\right)[u_s(t_2) - u_s(t'')] = 0$$
$$(r = 1, \ldots, p) \tag{36}$$

obtained from Eqs. (6) for $n = p+3$ with t'' playing the role of t_1. Note that m_{rs} $(r, s = 1, \ldots, p)$ include contributions from C (but not from C'). By reference to Eqs. (7) and (32), one has

$$m_{rs} = 0 \qquad (r = 1, \ldots, p;\ s = p+1, p+2, p+3) \tag{37}$$

$$m_{ks} = -\Delta m \delta_{ks} \qquad (k = p+1, p+2, p+3;\ s = 1, \ldots, p+3) \tag{38}$$

so that, in view of Eqs. (30), Eqs. (36) can be simplified, i.e.,

$$\sum_{s=1}^{p} m_{rs}[u_s(t_2) - u_s(t_1)] - \Delta m \sum_{s=p+1}^{p+3} \bar{C}_{sr}[u_s(t_2) - u_s(t_1)] = 0$$
$$(r = 1, \ldots, p) \tag{39}$$

and Eqs. (33) can be replaced with

$$u_k(t_1) = \sum_{r=1}^{p} \bar{C}_{kr} u_r(t_1) + \bar{D}_k \qquad (k = p+1, p+2, p+3) \tag{40}$$

Using the definitions of matrices $\bar{C}$, $\bar{E}$ [with $\boldsymbol{d}_i = \boldsymbol{n}_i$ $(i = 1, 2, 3)$], M, and u preceding Eq. (28), one can replace Eqs. (40), (34), and (39) with the matrix equations

$$\bar{u}(t_1) = \bar{C}u(t_1) + \bar{D} \tag{41}$$

$$\bar{u}(t_2) = \bar{C}u(t_2) + \bar{D} + \bar{E} \tag{42}$$

$$M[u(t_2) - u(t_1)] - \Delta m \bar{C}^T[\bar{u}(t_2) - \bar{u}(t_1)] = 0 \tag{43}$$

with $\bar{D}$ and $\bar{u}$ as 3×1 matrices having $\bar{D}_k$ and u_k $(k = p+1, p+2, p+3)$ as their elements in row $k - p$ $(k = p+1, p+2, p+3)$. Equation (28) appearing in step 4 is obtained when $\bar{u}(t_1)$ and $\bar{u}(t_2)$ are eliminated from Eq. (43) with the aid of Eqs. (41) and (42).

Last, note that Eqs. (26) and (27) are similar to the first and the third of Eqs. (35), respectively, with $\boldsymbol{d}_i$ replacing $\boldsymbol{n}_i$ $(i = 1, 2, 3)$. Such a replacement is in order because the entries of $\bar{C}^T\bar{C}$ and of $\bar{C}^T\bar{E}$ in Eq. (28) are all scalar products of the vectors ${}^N\boldsymbol{v}_1^{\bar{P}}, \ldots, {}^N\boldsymbol{v}_p^{\bar{P}}, {}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}$, and hence are independent of the reference frame in which these vectors are expressed. Step 4, and hence the entire procedure, thus is established.

Suppose that particles yet to be expelled are arranged as $\bar{v}$ rigid bodies $C_1, \ldots, C_{\bar{v}}$ of masses $m_1, \ldots, m_{\bar{v}}$ fixed in reference frames $A_1, \ldots, A_{\bar{v}}$ of S; that rigid bodies $C_1', \ldots, C_{\bar{v}}'$ of masses $\Delta m_1, \ldots, \Delta m_{\bar{v}}$ are released from $C_1, \ldots, C_{\bar{v}}$ in a manner similar to C' being released from C; that $\bar{v}$ particles $\bar{Q}_1, \ldots, \bar{Q}_{\bar{v}}$ of masses $\Delta m_1, \ldots, \Delta m_{\bar{v}}$ are simultaneously attached to $\bar{v}$ points $\bar{P}_1, \ldots, \bar{P}_{\bar{v}}$ and ejected from $\bar{P}_1, \ldots, \bar{P}_{\bar{v}}$ of S with relative velocities ${}^N\boldsymbol{v}^{\bar{Q}_1/\bar{P}_1}, \ldots, {}^N\boldsymbol{v}^{\bar{Q}_{\bar{v}}/\bar{P}_{\bar{v}}}$, respectively; and that $\bar{C}_{(q)}$ and $\bar{E}_{(q)}$ are matrices defined by reference to ${}^N\boldsymbol{v}^{\bar{Q}_q}$ and ${}^N\boldsymbol{v}^{\bar{Q}_q/\bar{P}_q}$ $(q = 1, \ldots, \bar{v})$ as are $\bar{C}$ and $\bar{E}$ defined by reference to ${}^N\boldsymbol{v}^{\bar{Q}}$ and ${}^N\boldsymbol{v}^{\bar{Q}/\bar{P}}$. Then, Eq. (43) is replaced with

$$M[u(t_2) - u(t_1)] - \sum_{q=1}^{\bar{v}} \Delta m_q \bar{C}_{(q)}^T\left[\bar{u}_{(q)}(t_2) - \bar{u}_{(q)}(t_1)\right] = 0$$

an equation leading to

$$u(t_2) = u(t_1) + \left[M - \sum_{q=1}^{\bar{v}} \Delta m_q \bar{C}_{(q)}^T \bar{C}_{(q)}\right]^{-1} \sum_{q=1}^{\bar{v}} \Delta m_q \bar{C}_{(q)}^T \bar{E}_{(q)} \tag{44}$$

Then the five-step procedure can be used to simulate motions of S if Eq. (44) replaces Eq. (28), and if equations similar to Eqs. (19–22) are written for each of $C_1, \ldots, C_{\bar{v}}$, and used as when $\bar{v} = 1$. Equations (23) and (24) apply, provided that $F_r + F_r^*$ and $(F_r + F_r^*)$ $(r = 1, \ldots, p)$ include contributions from $C_1, \ldots, C_{\bar{v}}$ and from $C_1', \ldots, C_{\bar{v}}'$, respectively.

Concerning case 2, suppose that $\bar{Q}$ is captured between t_1 and t_2; introduce t' and t'' as in Eq. (29) and regard the capture of $\bar{Q}$ as comprising three events: the arrest of $\bar{Q}$ at $\bar{P}$ between t_1 and t', the release of $\bar{Q}$ from $\bar{P}$ between t' and t'', and the redistribution of the mass of $\bar{Q}$ as a rigid body C', attached to C between t'' and t_2. Assuming that ${}^N\boldsymbol{v}^{\bar{Q}}(t_1)$ is known, and that ${}^N\boldsymbol{v}^{\bar{Q}}(t < t_1)$ can be expressed as in Eq. (32), one may conclude that $u_k(t_1)$ $(k = p+1, p+2, p+3)$ are known. Furthermore, the arrest of $\bar{Q}$ at $\bar{P}$ qualifies as hard imposition of constraints; hence, ${}^N\boldsymbol{v}^{\bar{Q}}(t') = {}^N\boldsymbol{v}^{\bar{P}}(t')$, as indicated by Eq. (17), with t' playing the role of t_2. Also, ${}^N\boldsymbol{v}^{\bar{P}}$ can be expressed as in Eq. (25); therefore,

$$u_k(t') = \sum_{r=1}^{p} \bar{C}_{kr} u_r(t') + \bar{D}_k \qquad (k = p+1, p+2, p+3) \tag{45}$$

where $\bar{C}_{kr}$ and $\bar{D}_k$ $(k = p+1, p+2, p+3;\ r = 1, \ldots, p)$ are defined as in Eqs. (35), and Eq. (43) is valid with t' replacing t_2. In fact, such a replacement need not actually be carried out: The release of $\bar{Q}$ from $\bar{P}$ $(t' \le t \le t'')$ and the attachment of C' to C $(t'' \le t \le t_2)$ qualify as soft removal and soft imposition of constraints, respectively, involving no changes in $u_1, \ldots, u_n$. Consequently, Eq. (43) remains intact, and t' in Eqs. (45) is replaced with t_2. Written in a matrix form, Eqs. (45) read

$$\bar{u}(t_2) = \bar{C}u(t_2) + \bar{D} \tag{46}$$

where $\bar{C}$, $\bar{D}$, u, and $\bar{u}$ are defined as in connection with Eqs. (41–43). Consequently, $\bar{u}(t_2)$ can be eliminated from Eq. (43), which now can be solved for $u(t_2)$, yielding

$$u(t_2) = [M - \Delta m \bar{C}^T \bar{C}]^{-1}[Mu(t_1) - \Delta m \bar{C}^T \bar{u}(t_1) + \Delta m \bar{C}^T \bar{D}] \tag{47}$$

Thus, an ongoing process of the capture of particles represented by $\bar{Q}$ can be simulated with the aid of the five-step procedure, with Eqs. (23) and the first p of Eqs. (24) exchanging roles in steps 2, 3, and 5 [then, the negative signs in Eqs. (19) and (21) are replaced with positive signs] and Eq. (47) replacing Eq. (28) in step 4. Finally, the five-step procedure applies to S if $\bar{v}$ particles are arrested simultaneously at $\bar{v}$ points $\bar{P}_1, \ldots, \bar{P}_{\bar{v}}$ of S, building up $\bar{v}$ rigid bodies $C_1, \ldots, C_{\bar{v}}$; if Eq. (47) gives way to

$$u(t_2) = \left[M - \sum_{q=1}^{\bar{v}} \Delta m_q \bar{C}_{(q)}^T \bar{C}_{(q)} \right]^{-1} \times \left\{ Mu(t_1) - \sum_{q=1}^{\bar{v}} \Delta m_q \bar{C}_{(q)}^T \left[\bar{u}_{(q)}(t_1) - \bar{D}_{(q)} \right] \right\} \tag{48}$$

where $\bar{C}_{(q)}$, $\bar{D}_{(q)}$, u, $\bar{u}_{(q)}$, and $\Delta m_{(q)}$ are defined in connection with $\bar{Q}_q$ as are $\bar{C}$, $\bar{D}$, u, $\bar{u}$, and Δm in connection with $\bar{Q}$ when $\bar{v} = 1$; and if Eqs. (23) and (24) are interpreted accordingly.

At times, C can be regarded as a particle located at P. Then, the governing equations of motions can be obtained from those associated with cases 1 and 2, respectively, if $I_i^C = 0$ $(i = 1, 2, 3)$ and $\boldsymbol{r}^{PC^*} = 0$ and if Eqs. (20–22) are disregarded.

Rigid Body Ejecting Two Particles

Let C be a rigid body of mass m and central moment of inertia I for $\boldsymbol{n}_3$, undergoing an unconstrained motion in a plane perpendicular to $\boldsymbol{n}_3$, a unit vector fixed in N; and define u_i $(i = 1, 2, 3)$ such that ${}^N\boldsymbol{v}^{C^*}$, the velocity in N of C^*, the mass center of C, and ${}^N\omega^C$, the angular velocity of C in N, are given by

$${}^N\boldsymbol{v}^{C^*} = u_1 \boldsymbol{n}_1 + u_2 \boldsymbol{n}_2, \qquad {}^N\omega^C = u_3 \boldsymbol{n}_3 \tag{49}$$

Hence, $p = 3$. The equations of motion of C in N and the associated mass matrix are

$$-m\dot{u}_1 = 0, \qquad -m\dot{u}_2 = 0, \qquad -I\dot{u}_3 = 0 \tag{50}$$

$$M = \begin{vmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -I \end{vmatrix} \tag{51}$$

Let $\bar{P}_1$ and $\bar{P}_2$ be two points of C whose position vectors relative to C^* are given by $\boldsymbol{p}^{\bar{P}_1} = -a\boldsymbol{c}_1 + b\boldsymbol{c}_2$ and $\boldsymbol{p}^{\bar{P}_2} = -a\boldsymbol{c}_1 - b\boldsymbol{c}_2$, where a and b are constants, and $\boldsymbol{c}_i$ $(i = 1, 2, 3)$ are three dextral, mutually perpendicular unit vectors fixed in C such that $\boldsymbol{c}_3 = \boldsymbol{n}_3$. Let C eject two particles $\bar{Q}_1$ and $\bar{Q}_2$ of masses m_1 and m_2 from $\bar{P}_1$ and $\bar{P}_2$, respectively, with relative velocities given by

$$\begin{aligned} {}^N\boldsymbol{v}^{\bar{Q}_1/\bar{P}_1} &= -v_1(\cos\alpha_1 \boldsymbol{c}_1 + \sin\alpha_1 \boldsymbol{c}_2) \\ {}^N\boldsymbol{v}^{\bar{Q}_2/\bar{P}_2} &= -v_2(\cos\alpha_2 \boldsymbol{c}_1 + \sin\alpha_2 \boldsymbol{c}_2) \end{aligned} \tag{52}$$

where v_1, v_2, α_1, and α_2 are constants. It is required that the associated changes in u_1, u_2, and u_3 be evaluated.

The velocities of $\bar{P}_1$ and $\bar{P}_2$ in N are given by

$$\begin{aligned} {}^N\boldsymbol{v}^{\bar{P}_1} &= u_1 \boldsymbol{n}_1 + u_2 \boldsymbol{n}_2 + u_3 \boldsymbol{n}_3 \times (-a\boldsymbol{c}_1 + b\boldsymbol{c}_2) \\ {}^N\boldsymbol{v}^{\bar{P}_2} &= u_1 \boldsymbol{n}_1 + u_2 \boldsymbol{n}_2 + u_3 \boldsymbol{n}_3 \times (-a\boldsymbol{c}_1 - b\boldsymbol{c}_2) \end{aligned} \tag{53}$$

In accordance with Eqs. (22) and (23) written for $r = 3$ and $k = 3$, one has

$$\bar{C}_{(1)} = \begin{vmatrix} \cos q_3 & \sin q_3 & -b \\ -\sin q_3 & \cos q_3 & -a \\ 0 & 0 & 0 \end{vmatrix}, \qquad \bar{E}_{(1)} = \begin{vmatrix} -v_1 \cos\alpha_1 \\ v_1 \sin\alpha_1 \\ 0 \end{vmatrix}$$
$$\bar{C}_{(2)} = \begin{vmatrix} \cos q_3 & \sin q_3 & b \\ -\sin q_3 & \cos q_3 & -a \\ 0 & 0 & 0 \end{vmatrix}, \qquad \bar{E}_{(2)} = \begin{vmatrix} -v_2 \cos\alpha_2 \\ v_2 \sin\alpha_2 \\ 0 \end{vmatrix} \tag{54}$$

Equation (44), written for $q = 1, 2$, reads

$$\begin{aligned} |u_1(t_2) \quad u_2(t_2) \quad u_3(t_2)|^T &= |u_1(t_1) \quad u_2(t_1) \quad u_3(t_1)|^T \\ &+ \left[M - m_1 \bar{C}_{(1)}^T \bar{C}_{(1)} - m_2 \bar{C}_{(2)}^T \bar{C}_{(2)} \right]^{-1} \\ &\times \left[m_1 \bar{C}_{(1)}^T \bar{E}_{(1)} + m_2 \bar{C}_{(2)}^T \bar{E}_{(2)} \right] \end{aligned} \tag{55}$$

where $\Delta m_i = m_i$ $(i = 1, 2)$. Substituting from Eqs. (51) and (54), and using the definitions

$$J \hat{=} I + (m_1 + m_2)(a^2 + b^2), \qquad m_t \hat{=} m + m_1 + m_2$$
$$x \hat{=} (m_1 + m_2)p \cos q_3 + (m_1 - m_2)r \sin q_3$$
$$y \hat{=} -(m_1 + m_2)a \sin q_3 + (m_1 - m_2)b \cos q_3$$
$$z \hat{=} m_1 v_1 (b \cos\alpha_1 - a \sin\alpha_1) - m_2 v_2 (b \cos\alpha_2 - a \sin\alpha_2)$$
$$v \hat{=} m_1 v_1 \cos(\alpha_1 - q_3) + m_2 v_2 \cos(\alpha_2 - q_3)$$
$$w \hat{=} m_1 v_1 \sin(\alpha_1 - q_3) + m_2 v_2 \sin(\alpha_2 - q_3)$$
$$s \hat{=} J m_t - x^2 - y^2$$

one can express $u_r(t_2)$ $(r = 1, 2, 3)$ as

$$\begin{aligned} u_1(t_2) &= u_1(t_1) + [v(s + y^2) - wxy]/m_t s - yz/s \\ u_2(t_2) &= u_2(t_1) - [w(s + x^2) - vxy]/m_t s - xz/s \\ u_3(t_2) &= u_3(t_1) - (m_t z + wx - vy)/s \end{aligned} \tag{56}$$

Next, suppose that $\bar{Q}_1$ and $\bar{Q}_2$ move independently of C prior to time t_1, and that at t_1 the velocities of $\bar{Q}_1$ and $\bar{Q}_2$ are given by

$$\begin{aligned} {}^N\boldsymbol{v}^{\bar{Q}_1} &= -v_1(\cos\alpha_1 \boldsymbol{c}_1 + \sin\alpha_1 \boldsymbol{c}_2) \\ {}^N\boldsymbol{v}^{\bar{Q}_2} &= -v_2(\cos\alpha_2 \boldsymbol{c}_1 + \sin\alpha_2 \boldsymbol{c}_2) \end{aligned} \tag{57}$$

Furthermore, suppose that when $t = t_1$, $\bar{Q}_1$ and $\bar{Q}_2$ impact $\bar{P}_1$ and $\bar{P}_2$ of C, respectively, and subsequently are arrested at $\bar{P}_1$ and $\bar{P}_2$. As before, it is required that the associated changes in u_1, u_2, and u_3 be evaluated.

Now, in accordance with Eqs. (57),

$$\begin{aligned} \bar{u}_{(1)} &= |-v_1 \cos\alpha_1 \quad v_1 \sin\alpha_1 \quad 0|^T \\ \bar{u}_{(2)} &= |-v_2 \cos\alpha_2 \quad v_2 \sin\alpha_2 \quad 0|^T \end{aligned} \tag{58}$$

Moreover, $\bar{D}_{(1)} = \bar{D}_{(2)} = 0$ [see Eqs. (53)], and $\bar{C}_{(1)}$, $\bar{C}_{(2)}$, and M remain as in Eqs. (51) and (54). Consequently, substitutions in Eq. (48) lead to

$$\begin{aligned} u_1(t_2) &= \left\{ [mu_1(t_1) - v](s + y^2) + [mu_2(t_1) + w]xy \right\} \big/ m_t s \\ &\quad + y[Iu_2(t_1) + z]/s \\ u_2(t_2) &= \left\{ [mu_2(t_1) + w](s + x^2) + [mu_1(t_1) - v]xy \right\} \big/ m_t s \\ &\quad + x[Iu_3(t_1) + z]/s \\ u_3(t_2) &= \left\{ m_t[Iu_3(t_1) + z] + [mu_2(t_1) + w]x \right. \\ &\quad \left. + [mu_1(t_1) - v]y \right\} \big/ s \end{aligned} \tag{59}$$

where the definitions of x, y, z, v, w, and s remain as before.

Equations (56) and (59) can be obtained by a direct application of momentum principles. However, these principles furnish only six equations, which are not sufficient to deal with systems possessing more than six degrees of freedom. Such is the variable-mass flexible rocket discussed in the next example.

Continuous and Discrete Particle-Ejecting Systems

The model of the rocket system S, a representative of continuous-particle-ejecting systems, is shown in Fig. 3. The system consists of a cylindrical, rigid body C of length L_C, fixed in A, a reference frame belonging to S, so that the axis of symmetry of C coincides with l, a line fixed in A; and of a cylindrical, uniform, deformable body B of length L_B, whose axis of symmetry coincides with l when undeformed. Let $\boldsymbol{a}_i$ $(i = 1, 2, 3)$ be three dextral, mutually perpendicular unit vectors fixed in A such that $\boldsymbol{a}_1$ is aligned with l and $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ are parallel to a plane p common to B and C. Let C, representing a solid-state motor, release particles that are simultaneously attached to $\bar{P}$ and ejected from $\bar{P}$, the expulsion point. Point $\bar{P}$ is fixed in A, and is located relative to P, the intersection point of line l and plane p (see Fig. 3), such that

$$\boldsymbol{r}^{P\bar{P}} = -L_C\boldsymbol{a}_1 + R\boldsymbol{a}_3 \tag{60}$$

where R is a constant called radial thrust (or jet) misalignment. Let $\bar{Q}$ be a typical particle being ejected from $\bar{P}$, and let the relative velocity of $\bar{Q}$ with respect to $\bar{P}$ be

$${}^A\boldsymbol{v}^{\bar{Q}/\bar{P}} = {}^N\boldsymbol{v}^{\bar{Q}/\bar{P}} = -v\cos\alpha\boldsymbol{a}_1 + v\sin\alpha\boldsymbol{a}_2 \tag{61}$$

where v, called the ejection speed, and α, called the angular thrust (or jet) misalignment, are known constants. It is required that the motion of S in N be simulated in an ongoing process in which particles are expelled from A at a rate of $\dot{m}$.

One may start with the introduction of six generalized speeds,

$$u_i \hat{=} {}^N\omega^A \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3) \tag{62}$$

$$u_{3+i} \hat{=} {}^N\boldsymbol{v}^P \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3) \tag{63}$$

where ${}^N\omega^A$ and ${}^N\boldsymbol{v}^P$ are, respectively, the angular velocity of A in N, and the velocity of point P of A in N. Let p_i $(i = 1, 2, 3)$ be constants defined such that

$$\boldsymbol{r}^{PC^*} = p_1\boldsymbol{a}_1 + p_2\boldsymbol{a}_2 + p_3\boldsymbol{a}_3 \tag{64}$$

where $\boldsymbol{r}^{PC^*}$ is the position vector form P to C^*, the mass center of C. Moreover, based on Eqs. (60–64), write

$${}^N\omega^A (= {}^N\omega^C) = u_1\boldsymbol{a}_1 + u_2\boldsymbol{a}_2 + u_3\boldsymbol{a}_3 \tag{65}$$

$${}^N\boldsymbol{v}^P = u_4\boldsymbol{a}_1 + u_5\boldsymbol{a}_2 + u_6\boldsymbol{a}_3 \tag{66}$$

$$\begin{aligned}{}^N\boldsymbol{v}^{C^*} &= u_4\boldsymbol{a}_1 + u_5\boldsymbol{a}_2 + u_6\boldsymbol{a}_3 + (u_2p_3 - u_3p_2)\boldsymbol{a}_1\\ &+ (u_3p_1 - u_1p_3)\boldsymbol{a}_2 + (u_1p_2 - u_2p_1)\boldsymbol{a}_3\end{aligned} \tag{67}$$

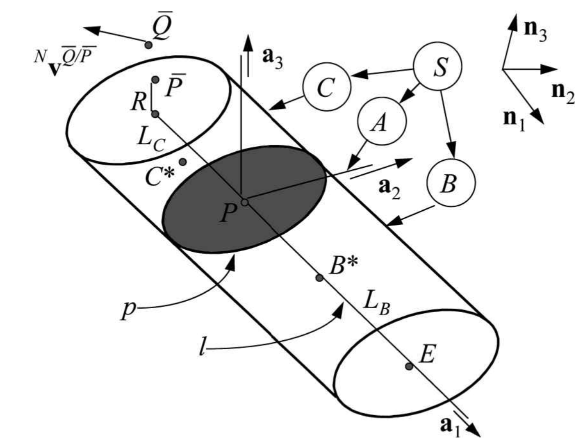

Fig. 3 Rigid-body model, with *C* representing a solid-state motor.

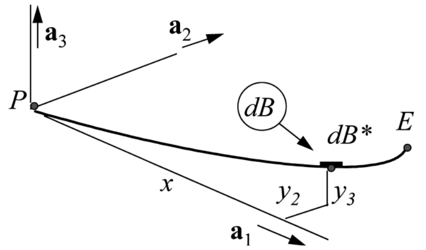

Fig. 4 Deformed configuration of neutral axis of *B*.

$${}^N\boldsymbol{v}^{\bar{P}} = u_4\boldsymbol{a}_1 + u_5\boldsymbol{a}_2 + u_6\boldsymbol{a}_3 + u_2R\boldsymbol{a}_1 - (u_3L_C + u_1R)\boldsymbol{a}_2 + u_2L_C\boldsymbol{a}_3 \tag{68}$$

With these in hand, one can form $(F_r + F_r^*)^C = 0$ $(r = 1, \ldots, 6)$, contributions of C to the sums of generalized active forces and of generalized inertial forces, disregarding momentarily the ejection of particles from C. One then uses m_C, I_1^C and C_{ij} $(i, j = 1, 2, 3)$, quantities appearing in Eqs. (19), (21), and (22).

Next, suppose B is capable of bending in A as shown in Fig. 4. Let $\mathrm{d}B$ be an element of length $\mathrm{d}x$ of the beam; let $\mathrm{d}B^*$ be the mass center of $\mathrm{d}B$, and let $\mathrm{d}B^*$ be located along the neutral axis of the beam at a distance x from P when the beam is undeformed. Let y_2 and y_3 be the elastic deflection of $\mathrm{d}B^*$, and let $\boldsymbol{b}_i$ $(i = 1, 2, 3)$ be three dextral, mutually perpendicular unit vectors fixed in $\mathrm{d}B$. To describe the orientation of $\mathrm{d}B$ in A, let $\boldsymbol{b}_i$ $(i = 1, 2, 3)$ be initially aligned with $\boldsymbol{a}_i$ $(i = 1, 2, 3)$, respectively; rotate $\mathrm{d}B$ an amount y_2' about $\boldsymbol{a}_3$, and then an amount $-y_3'$ about $\boldsymbol{b}_2$, where $(\,)'$ indicates differentiation with respect to x. Consequently, the direction cosines between $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ $(i = 1, 2, 3)$ are

	$\boldsymbol{b}_1$	$\boldsymbol{b}_2$	$\boldsymbol{b}_3$
$\boldsymbol{a}_1$	$\cos y_2' \cos y_3'$	$-\sin y_2'$	$-\cos y_2' \sin y_3'$
$\boldsymbol{a}_2$	$\sin y_2' \cos y_3'$	$\cos y_2'$	$-\sin y_2' \sin y_3'$
$\boldsymbol{a}_3$	$\sin y_3'$	0	$\cos y_3'$

(69)

One now can express the position of $\mathrm{d}B^*$ relative to P and the angular velocity of $\mathrm{d}B$ in A as follows:

$$\boldsymbol{p}^{P\,\mathrm{d}B^*} = x\boldsymbol{a}_1 + y_2\boldsymbol{a}_2 + y_3\boldsymbol{a}_3 \tag{70}$$

$${}^A\omega^{\mathrm{d}B} = -\dot{y}_2'\sin(-y_3')\boldsymbol{b}_1 - \dot{y}_3'\boldsymbol{b}_2 + \dot{y}_2'\cos(-y_3')\boldsymbol{b}_3 \tag{71}$$

where $(\dot{\ })$ indicates differentiation with respect to time. Moreover, in view of Eqs. (65) and (66), the velocity of $\mathrm{d}B^*$ in N and the angular velocity of $\mathrm{d}B$ in N are given by

$$\begin{aligned}{}^N\boldsymbol{v}^{\mathrm{d}B^*} &= {}^N\boldsymbol{v}^P + \frac{{}^N\mathrm{d}p^{P\,\mathrm{d}B^*}}{\mathrm{d}t} = u_4\boldsymbol{a}_1 + u_5\boldsymbol{a}_2\\ &+ u_6\boldsymbol{a}_3 + \dot{y}_1\boldsymbol{a}_2 + \dot{y}_3\boldsymbol{a}_3 + (u_2y_3 - u_3y_2)\boldsymbol{a}_1\\ &+ (u_3x - u_1y_3)\boldsymbol{a}_2 + (u_1y_2 - u_2x)\boldsymbol{a}_3\end{aligned} \tag{72}$$

$$\begin{aligned}{}^N\omega^{\mathrm{d}B} &= {}^N\omega^A + {}^A\omega^{\mathrm{d}B} = u_1\boldsymbol{a}_1 + u_2\boldsymbol{a}_2 + u_3\boldsymbol{a}_3\\ &+ \dot{y}_2'\sin(y_3')\boldsymbol{b}_1 - \dot{y}_3'\boldsymbol{b}_2 + \dot{y}_2'\cos(y_3')\boldsymbol{b}_3\end{aligned} \tag{73}$$

Now, in accordance with the assumed mode method, y_2 and y_3 can be expressed as

$$y_2 = \sum_{i=7}^{6+v} \phi_{i-6}q_i, \qquad y_3 = \sum_{i=7+v}^{6+2v} \phi_{i-6-v}q_i \tag{74}$$

where $\phi_1, \ldots, \phi_v$ are modal functions, $q_7, \ldots, q_{6+2v}$ are modal coordinates, and v is the number of modes used to describe elastic deformations of B; and where the same modal functions are used in the description of both y_2 and y_3 because of the rotational symmetry of B. Defining, in addition to $u_1, \ldots, u_6$, $2v$ generalized speeds as

$$u_{6+r} \hat{=} \dot{q}_{6+r} \qquad (r = 1, \ldots, 2v) \tag{75}$$

one can write expressions for $6 + 2v$ generalized inertial forces for dB. Furthermore, these expressions comprise functions of x that can be integrated over x (from 0 to L_B), and added to $6+2v$ expressions for the generalized active forces for B. The latter are given by

$$\begin{aligned} F_r &= 0 \qquad (r = 1, \ldots, 6) \\ F_r &= -EJL_B q_r \lambda_{r-6}^4 \qquad (r = 7, \ldots, 6 + v) \\ F_r &= -EJL_B q_r \lambda_{r-6-v}^4 \qquad (r = 7 + v, \ldots, 6 + 2v) \end{aligned} \tag{76}$$

where EJ is the flexible rigidity of B and $\lambda_1, \ldots, \lambda_v$ are eigenvalues associated with $\phi_1, \ldots, \phi_v$, respectively. The contributions from B to the sums of generalized active forces and generalized inertial forces $(F_r + F_r^*)^B = 0$ $(r = 1, \ldots, 6 + 2v)$ now can be formed and added to the respective contributions from C if note is taken that $(F_r + F_r^*)^C \equiv 0$ $(r = 7, \ldots, 6 + 2v)$. Setting $(F_r + F_r^*)^C + (F_r + F_r^*)^B = 0$ $(r = 1, \ldots, 6+2v)$, one obtains a set of equations of motion for the entire system. These are not reported here in full because of space limitations.

Finally, one has, in view of Eqs. (25), (26), and (27), (61) and (68),

$$\bar{C} = \begin{vmatrix} 0 & R & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -R & 0 & -L_C & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_C & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \tag{77}$$

$$\bar{E} = |-v\cos\alpha \quad v\sin\alpha \quad 0|$$

One is now in a position to use Eqs. (23), (24), and (28) in conjunction with the five-step procedure in simulating motions of S in which particles are expelled from S, that is, released from C and, simultaneously, attached at $\bar{P}$ and ejected from $\bar{P}$. Letting $m_B = 0.008$, $m_C = 0.0002 - 0.0002t$ kg $\cdot$ mm^{-1} $\cdot$ s^2, $I_1^B = 30$, $I_2^B = I_3^B = 2700$, $I_1^C = 0.6 - 0.6t$, $I_2^C = I_3^C = 4 - 4t$ kg $\cdot$ mm $\cdot$ s^2, $L_B = 2000$, $L_C = 500$, $R = 0.4$ mm, $\alpha = 0.01$ rad, $v = 1.8 \times 10^6$ mm/s, $p_1 = -250 + 5\sin\pi t$, $p_2 = p_3 = 3\sin 2\pi t$ mm, $C_{11} = \cos\varphi$, $C_{12} = -\sin\varphi$, $C_{13} = 0$, $C_{21} = \sin\varphi\cos\omega t$, $C_{22} = \cos\varphi\cos\omega t$, $C_{23} = -\sin\omega t$, $C_{31} = \sin\varphi\sin\omega t$, $C_{32} = \cos\varphi\sin\omega t$, $C_{33} = \cos\omega t$, and $\varphi = 2$ deg, $EJ = 2.1 \times 10^9$ kg $\cdot$ mm^2, and $v = 3$; and using modal functions associated with planar motions of B (regarded as a Bernoulli–Euler beam) in A when A is fixed in N, one obtains Fig. 5, showing u_1, u_2, and u_3 as functions of time, and Fig. 6, showing $y_2(L)$ and $y_3(L)$ as functions of time.

Next, the force replacement approach is applied to the problem at hand. Accordingly, it is assumed that a force $\boldsymbol{F}^{\bar{P}}$ given by

$$\boldsymbol{F}^{\bar{P}} = -\dot{m}_C {}^N\boldsymbol{v}^{\bar{Q}/\bar{P}} \tag{78}$$

and exerted on $\bar{P}$ has the same effect on the motion of the system as does the particle expulsion. Suppose that this force is regarded as an active force applied at $\bar{P}$, and that the equations of motion of the system in question are modified to include contributions from $\boldsymbol{F}^{\bar{P}}$. Note that the five-step procedure, with the exception of step 4, takes care of the changes in the inertial properties of S following particle expulsion, and suppose that this procedure, with the exception of step 4, is applied to the modified equations. Then, results indistinguishable from those in Figs. 5 and 6 are obtained, provided the time step is taken to be 0.0001 s (or smaller). One may conclude that results obtained with the force replacement approach are identical to results obtained with the aid of the new algorithm, provided the integration time step is chosen appropriately.

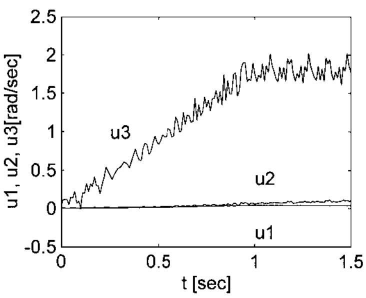

Fig. 5 Angular velocity measure numbers of a flexible rocket with thrust misalignments.

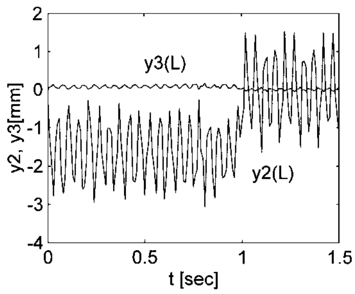

Fig. 6 Elastic deflection of point *E*.

A representative of discrete particle-ejecting systems may be an automatic gun firing projectiles weighing w_P kg at a rate of r rounds/min, with muzzle velocity v m/s. When the five-step procedure is used to simulate motions of the gun, step 4, and hence, Eq. (28), is invoked r times/min, giving rise to an instantaneous change in $u_1, \ldots, u_p$ every $1/r$ min; and $\dot{m}_P \Delta t$ is taken to be w_P/g, where g is the gravitational constant. With reference to the force replacement approach, the magnitude of the replacing force cannot be taken to be $\dot{m}_P v = r(w_P/g)v$, as with continuous particle-ejecting systems. For, $\dot{m}_P$ represents particle flow only if the flow is continuous in the sense that at each integration step Δt a particle of mass $\dot{m}_P \Delta t$ is ejected, irrespective of the value of Δt. Thus, to proceed with the force replacement approach, one needs additional information concerning the time history of the force exerted on the gun with every round being fired.

Past experience can give one a fairly good idea of what this force would be, as can direct measurements. However, muzzle velocities are quantities routinely measured with great precision, and usually referred to when automatic weapons are discussed (see, e.g., Ref. 12). With these in hand, the new algorithm provides a comprehensive means for the simulation of motions of simple, nonholonomic systems expelling and/or capturing particles.

Conclusions

The five-step procedure represents an algorithm underlying programs capable of simulating motions of variable-mass systems. The associated equations cannot be used for analytical investigations. However, they are unique in more than one respect: They apply to simple, nonholonomic systems; they allow particles being expelled (captured) to be arranged—before (after) the expulsion (capture)—as a number of rigid bodies, each possessing time-dependent inertial properties; they allow complete separation between the release (attachment) part of the process and the ejection (arrest) part; and they involve two updates of the integration variables at each integration step: one associated with the numerical integration and another bringing into evidence the effect of the expulsion and/or capture of particles. Finally, the five-step procedure allows the simulation of motions of continuous and discrete variable-mass systems alike. However, if the forces exerted on the system during the expulsion (capture) are known, then the force-replacement approach is advantageous in that it introduces replacing forces that are treated similarly to other contact forces, and it can take into account the internal particle-flow effects disregarded by other methods.

References

[1]Meirovitch, L., "General Motion of a Variable Mass Flexible Rocket with Internal Flow," *Journal of Spacecraft and Rockets*, Vol. 7, No. 2, 1970, pp. 186–195.

[2]McPhee, J. J., and Dubey, R. N., "Dynamic Analysis and Computer Simulation of Variable-Mass Multi-Rigid-Body Systems," *International Journal for Numerical Methods in Engineering*, Vol. 32, Sept. 1991, pp. 1711–1725.

[3]Eke, F. O., and Wang, S. M., "Equations of Motion of Two-Phase Variable Mass Systems with Solid Base," *Journal of Applied Mechanics*, Vol. 61, No. 4, 1994, pp. 855–860.

[4]Shames, I. H., *Mechanics of Fluids*, 2nd ed., McGraw–Hill, New York, 1982, Chap. 4.

[5]Ge, Z. M., and Cheng, Y. H., "Extended Kane's Equations for Non-Holonomic Variable Mass System," *Journal of Applied Mechanics*, Vol. 49, June 1982, pp. 429–431.

[6]Ibrahim, A. M., and Modi, V. J., "A Formulation for Studying Dynamics of *n* Connected Flexible Deployable Members," *Acta Astronautica*, Vol. 16, Sept. 1987, pp. 151–164.

[7]Greensite, A. L., *Analysis and Design of Space Vehicles Flight Control System*, Spartan, New York, 1970, Secs. 4.2 and 4.3 and Appendix C.

[8]Cveticanin, L., "Conservation Laws in Systems with Variable Mass," *Journal of Applied Mechanics*, Vol. 60, Dec. 1993, pp. 954–958.

[9]Cveticanin, L., "Some Conservation Laws for Orbit Involving Variable Mass and Linear Damping," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 1, 1994, pp. 209–211.

[10]Djerassi, S., "Imposition of Constraints," *Journal of Applied Mechanics*, Vol. 61, No. 2, 1994, pp. 434–439.

[11]Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications*, McGraw–Hill, New York, 1985, Chap. 2.

[12]Cullen, T., and Foss, C. F. (eds.), *Jane's Land-Based Air Defense*, 9th ed., Jane's Information Group, Surrey, England, UK, 1996, p. 195.